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SMOOTH CLASSIFICATION OF GEOMETRICALLY FINITE ONE-DIMENSIONAL MAPS

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ABSTRACT. The scaling function of a one-dimensional Markov map is defined and studied. We prove that the scaling function of a non-critical geometrically finite one-dimensional map is Hölder continuous, while the scaling function of a critical geometrically finite one-dimensional map is discontinuous. We prove that scaling functions determine Lipschitz conjugacy classes, and moreover, that the scaling function and the exponents and asymmetries of a geometrically finite one-dimensional map are complete C^1-invariants within a mixing topological conjugacy class.

1. INTRODUCTION

Two smooth self-maps f and g of a one-dimensional compact C^2-Riemannian manifold M are topologically conjugate if there is a homeomorphism h from M onto itself such that $h \circ f = g \circ h$. In the paper [7], we proved that if f and g are both geometrically finite, then h is quasisymmetric [1], whence it is Hölder continuous [1]. Usually h is not smooth because f has a lot of smooth invariants, for example, all eigenvalues of f at periodic points. Let f and g be orientation-preserving, expanding circle endomorphisms. Shub and Sullivan [10] proved that if h is absolutely continuous, then it is smooth. Sullivan [11] also proved that if the eigenvalues of f and g at all corresponding periodic points are the same, then h is smooth. Similar work has been done by Herman [3] for circle diffeomorphisms and by de la Llave and R. Moriyón [8] for Anosov diffeomorphisms of a torus. All these are results for maps without critical points. An important problem is to smoothly classify maps with critical points. Our first result [4, 5] in this direction gives the smooth classification of Ulam-von Neumann transformations (certain folding maps of an interval with one power law critical point). We proved that the eigenvalues at all periodic points together with the exponent and the asymmetry at the unique critical point of an Ulam-von Neumann transformation are complete smooth invariants. In general, all eigenvalues at the periodic points of a map may not be enough in the smooth classification of maps. To overcome this difficulty, Sullivan [12] introduced from physics (see, for example, [2]) a slightly stronger smooth invariant, the scaling function of a Cantor set, to smoothly classify certain embedded Cantor sets in the real line. In this paper, we define the scaling function for a Markov map and study the scaling function for a geometrically finite one-dimensional map. We use scaling

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functions to smoothly classify geometrically finite one-dimensional maps within a topological conjugacy class.

Suppose M is a one-dimensional compact C^2-Riemannian manifold. A geometrically finite one-dimensional map f is a certain self-map of M with finitely many power law singular points. We give the definition of a geometrically finite one-dimensional map f in §3.1. We give the definition of the dual phase space Σ^* and of the scaling function s, on Σ^*, of a Markov map f with a fixed Markov partition η of M to f in §2.2. A singular point c of a geometrically finite one-dimensional map f is critical if f is differentiable at c and $f'(c) = 0$. A non-critical singular point is a non-differentiable point of f. A geometrically finite one-dimensional map f is said to be critical if it has a critical singular point and to be non-critical if it has only non-critical singular points (see Definition 4 in §3.1). For a geometrically finite one-dimensional map f, there is a natural Markov partition η of M to f which consists of all intervals in the complement of the orbits of singular points of f in M. We only consider this natural Markov partition η as a fixed Markov partition for a geometrically finite one-dimensional map f. Then we can say that the dual phase space Σ^* of a geometrically finite one-dimensional map f. A function s on Σ^* is said to be Hölder if there are constants $C > 0$ and $0 < \nu < 1$ such that $|s(a_1^*) - s(a_2^*)| \leq C\nu^n$ whenever the first n digits w_n of $a_1^* = \cdots w_n$ and $a_2^* = \cdots w_n$ in Σ^* are the same. In §3.2 and §3.3 we prove the existence of the scaling function s on Σ^* of a geometrically finite one-dimensional map f.

Theorem 1. *Let f be a non-critical geometrically finite one-dimensional map. Then the scaling function s on Σ^* of f exists and is Hölder.*

Theorem 2. *Let f be a critical geometrically finite one-dimensional map. Then the scaling function s, on Σ^*, of f exists.*

Furthermore, because of the existence of critical points for a critical geometrically finite one-dimensional map, we prove that

Corollary 1. *Let f be a critical geometrically finite one-dimensional map. Then its scaling function s is discontinuous on Σ^*.*

In §2.2, we prove that the scaling function of a Markov map f with a fixed Markov partition η of M to f is a stronger smooth invariant than the set of all eigenvalues at periodic points of f as follows: let Σ^* be the dual space of f with η, let σ^* be the shift map of Σ^*, and let h_* be a map from the set of periodic points of σ^* onto the set of periodic points of f, which we define in §2.2.

Proposition 3. *Let f be a Markov map and let η be a fixed Markov partition of M to f. Assume that the scaling function s on Σ^* of f with η exists. Then for every periodic point a^* of σ^* of period m,*

$$\frac{1}{|E_p|} = \prod_{l=0}^{m-1} s((\sigma^*)^{\circ l}(a^*)),$$

where $p = h_*(a^*)$ is a periodic point of f of period m and $E_p = (f^{\circ m})'(p)$ is the eigenvalue of f at p.

An object is said to be a C^1-invariant if it is the same for f and for $h \circ f \circ h^{-1}$ whenever h is a C^1-diffeomorphism of M.

Proposition 4. *Let f be a Markov map and let η be a fixed Markov partition of M to f. The scaling function s, on Σ^*, of f with η (if it exists) is a C^1-invariant.*

A geometrically finite one-dimensional map f is said to be simple if its post-critical orbits and its set of critical points are disjoint. To avoid notational complication, we restrict discussion in §3 to simple geometrically finite one-dimensional maps. Let $\mathcal{F}$ be an (orientation-preserving) topological conjugacy class in the space of simple geometrically finite one-dimensional maps; that is, $\mathcal{F}$ is a set of simple geometrically finite one-dimensional maps such that every pair f and g in $\mathcal{F}$ are topologically conjugate by an orientation-preserving homeomorphism h of M. Since all maps f in $\mathcal{F}$ have the same dual phase space Σ^*, we may speak of the dual phase space Σ^* of a topological conjugacy class $\mathcal{F}$. We note that a topological conjugacy class $\mathcal{F}$ can be described by a kneading sequence (see [9]). We prove in §3.4 that scaling functions determine Lipschitz conjugacy classes in $\mathcal{F}$ as follows:

Theorem 3. *Let f and g be maps in $\mathcal{F}$ and let h be the conjugacy from f to g. Then h is bi-Lipschitz continuous if the scaling functions s_f and s_g, on Σ^*, of f and g are the same.*

A singular point p of f is said to be a fold singular point if $f'(x)f'(2p-x) < 0$ for $x \neq p$ near p. Let FSP be the set of all fold singular points of a geometrically finite one-dimensional map f, and let $FSO = \bigcup_{i=1}^{\infty} f^{\circ i}(FSP)$ be the post fold singular orbits. Let ξ be the set of closures of intervals of $M \setminus FSO$. A geometrically finite one-dimensional map f is mixing if for any interval I in the natural Markov partition η, there is an integer $n > 0$ such that $f^{\circ n}(I) = M$. Since the mixing condition is a topological invariant, we may speak of a topological conjugacy class $\mathcal{F}$ being mixing. We will prove that within a mixing topological conjugacy class $\mathcal{F}$, the scaling function s, on Σ^*, and the exponents and asymmetries of a map f in $\mathcal{F}$ are complete C^1-invariants. We first prove a rigidity result for geometrically finite one-dimensional maps. Let GSO be the grand singular orbits of f (see §3.4).

Theorem 4. *Let f and g be maps in a mixing topological conjugacy class $\mathcal{F}$. Let h be the conjugacy from f to g, i.e., $h \circ f = g \circ h$. Then $h|I$ for any I in ξ is a $C^{1+\beta}$-diffeomorphism for some $0 < \beta \leq 1$ if and only if (i) h is differentiable at one point p in $M \setminus GSO$ with non-zero derivative and (ii) the exponents and asymmetries of f and g at all corresponding critical points are the same.*

Since a Lipschitz homeomorphism h is absolutely continuous, it is differentiable at almost all points with non-zero derivatives in M. Thus Theorems 3 and 4 tell us

Theorem 5. *Let f and g be maps in a mixing topological conjugacy class $\mathcal{F}$ and let Σ^* be the dual space of $\mathcal{F}$. Let h be the topological conjugacy from f to g, i.e., $f \circ h = h \circ g$. Then $h|I$ for any I in ξ is a $C^{1+\beta}$-diffeomorphism for some $0 < \beta \leq 1$ if and only if the scaling functions s_f and s_g, on Σ^*, and the exponents and asymmetries of f and g at corresponding critical points are the same.*

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2. The scaling structure of a Markov map

Let M be an oriented compact one-dimensional C^2-Riemannian manifold and let $f: M \to M$ be a continuous self-map. The map f is said to be Markov if there is a set $\eta = \{I_1, \cdots, I_k\}$ of closed intervals of M such that

a. $I_1, \cdots, I_k$ have pairwise disjoint interiors,

b. the union $\bigcup_{i=1}^{k} I_i$ of all intervals in η is M,

c. the restriction $f|I$ to every interval I in η is injective and continuous, and

d. the image $f(I)$ of every interval I in η under f is the union of some intervals in η.

Remark 1. A set of closed intervals satisfying **a**, **b**, **c**, and **d** is called a Markov partition of M to f. For a Markov map f, one can find many Markov partitions of M to f. But there is the natural one for a map which we discuss in §3.

2.1. **Symbolic dynamical systems and phase spaces.** Let f be a Markov map and let $\eta = \{I_1, \cdots, I_k\}$ be a fixed Markov partition of M to f. Let g_i be the inverse of the restriction $f|I_i$ for $i = 1, \cdots, k$. A sequence $w_n = i_0 \cdots i_{n-1}$ of symbols $\{1, \cdots, k\}$ is said to be admissible if $I_{i_m} \subset f(I_{i_{m-1}})$ for every $m = 1, \cdots, n-1$. For an admissible sequence $w_n = i_0 \cdots i_{n-1}$, define $g_{w_n} = g_{i_0} \circ \cdots \circ g_{i_{n-1}}$, and define $I_{w_n} = g_{w_n}(f(I_{i_{n-1}}))$. We use η_n to denote the set of intervals I_{w_n} for all admissible sequences w_n of length n; we call η_n the n^{th} partition of M induced from f with η. (It is also a Markov partition.) We use λ_n to denote the maximum of the lengths of intervals in η_n. We always assume that λ_n tends to zero as n goes to infinity. Then (Σ, σ) is the phase space of f with η, where $\Sigma = \{a = i_0 i_1 \cdots\}$ is the set of all infinite admissible sequences with the product topology and $\sigma(i_0 i_1 \cdots) = i_1 \cdots$ is the shift map of Σ.

Proposition 1. *Let f be a Markov map and let η be a fixed Markov partition. Then there is a continuous map h from Σ onto M such that $f \circ h = h \circ \sigma$ on Σ.*

Proof. For every $a = i_0 i_1 \cdots$ in Σ, let $w_n = i_0 i_1 \cdots i_{n-1}$. Every w_n is admissible and $I_{w_{n+1}} \subseteq I_{w_n}$ for $n > 0$. Hence $\bigcap_{n=0}^{\infty} I_{w_n}$ is a nonempty set. But from our assumption that λ_n tends to zero as n goes to infinity, this set contains only one number x_a. Set $h(a) = x_a$. Then

$$f(h(a)) = f(x_a) = f\left(\bigcap_{n=0}^{\infty} I_{w_n}\right) = \bigcap_{n=0}^{\infty} f(I_{w_n}) = \bigcap_{n=0}^{\infty} I_{\sigma(w_n)} = h(\sigma(a)).$$

Now let us show that h is continuous. Two points a and b in Σ are n-close if the first n digits of them are the same. Suppose $a = i_0 \cdots i_{n-1} i_n \cdots$ and $b = i_0 \cdots i_{n-1} i'_n \cdots$ are n-close. Then $h(a)$ and $h(b)$ are in the same interval I_{w_n}, where $w_n = i_0 \cdots i_{n-1}$. This implies that $|h(a) - h(b)| \leq \lambda_n$. So $h(b)$ tends to $h(a)$ in M as b tends to a in Σ because λ_n tends to zero as n goes to infinity. This means that h is continuous at every point a in Σ.

The map h is onto because $\bigcup_{I \in \eta_n} I = M$ for every $n \geq 0$. Actually, h is one-to-one except for countably many points which are the preimages, under h, of endpoints of intervals of η_n, for $n \geq 0$. ∎

The dynamical system σ of Σ is a topological model of a topological conjugacy class as follows:

Proposition 2. *Let f be a Markov map and let η be a fixed Markov partition of M to f. Suppose (Σ, σ) is the phase space of f with η. A Markov map g is topologically conjugate to f if and only if there is a Markov partition η' of M to g such that the phase space of g with η' is (Σ, σ).*

Proof. Suppose g is topologically conjugate to f. There is a homeomorphism H of M such that $H \circ f = g \circ H$. Let $\eta' = H(\eta)$. Then η' is a Markov partition of M to g and the phase space of g with η' is (Σ, σ).

Now suppose there is a Markov partition η' of M to g such that the phase space of g with η' is also (Σ, σ). From Proposition 1, there are two continuous maps h_1 and h_2 from Σ onto M such that $f \circ h_1 = h_1 \circ \sigma$ and $g \circ h_2 = h_2 \circ \sigma$. Let $H = h_2 \circ h_1^{-1}$. It is defined on M except for countably many points which are endpoints of intervals in the partitions $\{\eta_n\}$ under h_1. The map H is also uniformly continuous. So it can be extended to a continuous map from M to M. Using the same argument, $H^{-1} = h_1 \circ h_2^{-1}$ can be also extended to a continuous map from M to M. Hence H is a homeomorphism of M. ∎

2.2. **Dual symbolic spaces and scaling functions.** Suppose f is a Markov map and $\eta = \{I_1, \cdots, I_k\}$ is a fixed Markov partition of M to f. In this section, we consider another symbolic space induced from f with η.

Let Γ_n be the set of all admissible sequences (see §2.1) w_n of length n. An (n,m)-right cylinder for $0 \leq m \leq n-1$ is $\{w_n = i_{n-1} \cdots i_0 \in \Gamma_n \mid i_l = i_l^0, l = 0, \cdots, m\}$, where $w_n^0 = i_{n-1}^0 \cdots i_0^0$ is a fixed sequence in Γ_n. All the (n,m)-right cylinders form a topological basis of Γ_n. Let Γ_n^* be the set Γ_n with this topological basis and $\left(\Sigma^*, \sigma^*\right)$ be the inverse limit of the sequence $\{\left(\Gamma_n^*, I_n^*\right)\}_{n=0}^{\infty}$, where $I_n^* : \Gamma_n^* \to \Gamma_{n-1}^*$ is the inclusion and $\sigma^* : \Sigma^* \to \Sigma^*$ is the shift. We call $\Sigma^* = \{a^* = \cdots i_1 i_0\}$ the dual phase space of f with η. The scaling function of f with η is a function defined on the dual phase space Σ^*. For $a^* = \cdots i_1 i_0$ and $w_n = i_{n-1} \cdots i_1 i_0$, we have $\sigma^*(a^*) = \cdots i_1$; we also denote $\sigma^*(w_n)$ by $i_{n-1} \cdots i_1$.

Let $a^* = \cdots w_n$ in Σ^*. Define

$$s(w_n) = \frac{|I_{w_n}|}{|I_{\sigma^*(w_n)}|}.$$

Definition 1. If $\lim_{n \to +\infty} s(w_n)$ exists for every a^* in Σ^*, then we define the scaling function $s(a^*) = \lim_{n \to +\infty} s(w_n)$ on Σ^*.

Let a^* be a periodic point of σ^* of period m. Then $a^* = w_m^\infty = \cdots w_m \cdots w_m$ where $w_m = i_{m-1} \cdots i_0$ is an admissible sequence of length m. We may define $h_*(a^*) = \bigcap_{i=1}^{\infty} I_{w_m^i}$ since $I_{w_m^{i+1}} \subseteq I_{w_m^i}$; $h_*(a^*)$ is a periodic point of f of period m.

Proposition 3. *Let f be a Markov map and let η be a fixed Markov partition of M to f. Assume that the scaling function s on Σ^* of f with η exists. Then for every periodic point a^* of σ^* of period m,*

$$\frac{1}{|E_p|} = \prod_{l=0}^{m-1} s((\sigma^*)^{\circ l}(a^*)),$$

where $p = h_*(a^*)$ is a periodic point of f of period m and $E_p = (f^{\circ m})'(p)$ is the eigenvalue of f at p.

Proof. Suppose $a^* = w_m^\infty$. Then $f^{\circ m}(I_{w_m^{i+1}}) = I_{w_m^i}$. So

$$\left|(f^{\circ m})'(b_i)\right|^{-1} = \frac{|I_{w_m^{i+1}}|}{|I_{w_m^i}|} = \prod_{l=0}^{m-1} s((\sigma^*)^{\circ l}(w_m^{i+1}))$$

for some b_i in $I_{w_m^{i+1}}$. As i goes to infinity, b_i tends to p and $s((\sigma^*)^{\circ l}(w_m^{i+1}))$ tends to $s((\sigma^*)^{\circ l}(a^*))$. Hence

$$\left|(f^{\circ m})'(p)\right|^{-1} = \prod_{l=0}^{m-1} s((\sigma^*)^{\circ l}(a^*)).$$

■

Proposition 4. *Let f be a Markov map and let η be a fixed Markov partition of M to f. The scaling function s, on Σ^*, of f with η (if it exists) is a C^1-invariant.*

Proof. Let $g = h \circ f \circ h^{-1}$ where h is a C^1-diffeomorphism of M. Then g is a Markov map and $\eta' = h(\eta)$ is a Markov partition of M to g. Let s_f be the scaling function on Σ^* of f with η and let s_g be the scaling function s on Σ^* of g with η'.

For every $a^* = \cdots w_n$, w_n is admissible and

$$s_g(w_n) = \frac{|h(I_{w_n})|}{|h(I_{\sigma^*(w_n)})|} = \frac{|h'(b_n)|}{|h'(b_n')|} \cdot \frac{|I_{w_n}|}{|I_{\sigma^*(w_n)}|} = \frac{|h'(b_n)|}{|h'(b_n')|} \cdot s_f(w_n)$$

where b_n and b_n' are in I_{w_n}. As n goes to infinity, $s_g(w_n) \to s_g(a^*)$, $s_f(w_n) \to s_f(a^*)$, and $|h'(b_n)|/|h'(b_n')| \to 1$. Hence $s_g(a^*) = s_f(a^*)$. So $s_f = s_g$ on Σ^*. ■

3. Geometrically finite one-dimensional maps

Let M be a one-dimensional compact C^2-Riemannian manifold. Suppose that $f : M \to M$ is continuous and piecewise C^1. A singular point a of f is either a non-differentiable point or a differentiable point with zero derivative. A singular point a of f is said to be a power law singular point if there is a $\gamma \geq 1$ such that

$$\lim_{x \mapsto a+} \frac{f'(x)}{|x-a|^{\gamma-1}} \quad \text{and} \quad \lim_{x \mapsto a-} \frac{f'(x)}{|x-a|^{\gamma-1}}$$

exist with nonzero limits B_+ and B_-. The numbers γ and $A = B_+/B_-$ are called the exponent and asymmetry of f at a.

Remark 2. Both γ and A are orientation-preserving C^1-invariants, i.e., they are the same for f and $h \circ f \circ h^{-1}$ whenever h is an orientation-preserving C^1-diffeomorphism of M.

Henceforth, we assume that f has only power law singular points and, without loss of generality, that f maps the boundary of M (if not empty) into itself and that the one-sided derivatives of f at all boundary points of M are non-zero. We note that in the general case, a boundary point of M should count as a singular point anyway. We call a singular point with exponent $\gamma > 1$ a critical point. Remember that a singular point with exponent $\gamma = 1$ is a non-differentiable point of f and that a critical point is a differentiable point of f with zero derivative.

Let $NP = \{a_1, \cdots, a_{d'}\}$ be the set of non-differentiable points of f and let $CP = \{c_1, \cdots, c_d\}$ be the set of critical points of f. Let $\Gamma = \{\gamma_1, \cdots, \gamma_d\}$ be the set of corresponding exponents at critical points of f. Then $S = NP \cup CP$ is the set of singular points of f. Let $SO = \bigcup_{n=0}^\infty f^{\circ n}(S)$ be the set of orbits of singular points. Let $PCO = \bigcup_{n=1}^\infty f^{\circ n}(CP)$ be the set of post-critical orbits of f.

Definition 2. The map f is said to be $C^{1+\alpha}$ for some $0 < \alpha \leq 1$ if

i. for every interval I in the complement of S in M, $f|I$ is C^1 and $(f|I)'$ is α-Hölder continuous, and

ii. for every critical point c_i of f, there is an open neighborhood U_i of c_i such that $f'(x)/|x-c_i|^{\gamma_i - 1}$ is α-Hölder continuous when restricted to $\{x < c\} \cap U_i$ and to $\{x > c\} \cap U_i$.

3.1. **A geometrically finite one-dimensional map.** Let f be a continuous map from M into M. If the set SO of singular orbits is non-empty and finite, then f is a Markov map and there is a natural Markov partition η which consists of the closures of intervals in the complement of SO in M. Let η_n be the n^{th} partition of M induced from f with η and let λ_n be the maximum of the lengths of intervals in η_n.

Definition 3. A one-dimensional map $f : M \to M$ is said to be geometrically finite if

(1) f is $C^{1+\alpha}$,

(2) the set of singular orbits SO is non-empty and finite,

(3) no critical point is periodic, and

(4) there are constants $C > 0$ and $0 < \mu < 1$ such that $\lambda_n \leq C\mu^n$ for all integers $n \geq 0$.

For a geometrically finite one-dimensional map f, we always take the natural Markov partition η which is the set of the closures of intervals in the complement of SO in M, and assume that η_n is the n^{th} partition induced from f with η and that Σ^* is the dual phase space of f with η. Within this context, we can talk about the dual phase space and the scaling function of f. We can also fix an integer $n_0 > 0$ such that the closure $\overline{U}_i$ of every open interval U_i in **ii** of Definition 2 is the union of two intervals in η_{n_0} and such that $U = \bigcup_{i=1}^{d} U_i$ is disjoint with $PCO \setminus CP$ (where $PCO = \bigcup_{i=1}^{\infty} f^{\circ i}(CP)$ is the set of post-critical orbits of f). Let V be the closure of the complement of U in M. These notations will be fixed for the rest of the paper.

Definition 4. A geometrically finite one-dimensional map f is said to be non-critical if it has no critical point; and f is said to be critical if it has critical points.

3.2. **A non-critical geometrically finite one-dimensional map.** Let f be a geometrically finite one-dimensional map and let Σ^* be the dual phase space of f. A function s on Σ^* is said to be Hölder if there are constants $C > 0$ and $0 < \nu < 1$ such that $|s(a_1^*) - s(a_2^*)| \leq C\nu^n$ whenever the first n digits of a_1^* and a_2^* in Σ^* are the same.

Theorem 1. *Let f be a non-critical geometrically finite one-dimensional map. Then the scaling function s on Σ^* of f exists and is Hölder.*

To prove Theorem 1, we first state and prove some important lemmas.

Lemma 1 (the naive distortion lemma)**.** *Suppose g from a set $W \subseteq M$ into M is a $C^{1+\alpha}$ map for some $0 < \alpha \leq 1$ and suppose $a_0 = \inf_{x \in W} |g'(x)| > 0$. Let $b_0 = \sup_{x \neq y \in W} \frac{|g'(x) - g'(y)|}{|x-y|^{\alpha}} < \infty$. Then for any two sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ of W,*

$$\Big|\log\Big(\prod_{i=1}^{n}\Big|\frac{g'(x_i)}{g'(y_i)}\Big|\Big)\Big| \leq \frac{b_0}{a_0}\sum_{i=1}^{n}|x_i - y_i|^{\alpha}.$$

Proof. The proof is easy since

$$\Big|\log\Big(\prod_{i=1}^{n}\Big|\frac{g'(x_i)}{g'(y_i)}\Big|\Big)\Big| \leq \sum_{i=1}^{n}\Big|\log|g'(x_i)| - \log|g'(y_i)|\Big|$$

$$\leq \sum_{i=1}^{n}\frac{1}{a_0}|g'(x_i) - g'(y_i)| \leq \sum_{i=1}^{n}\frac{b_0}{a_0}|x_i - y_i|^{\alpha}.$$

∎

Lemma 2. *Suppose f is a non-critical geometrically finite one-dimensional map. Then there are constants $C > 0$ and $0 < \mu < 1$ such that for any integers n, $m > 0$ and any interval I in η_{n+m},*

$$\max_{x,y\in I}\Big\{\Big|\log\Big(\frac{|(f^{\circ m})'(x)|}{|(f^{\circ m})'(y)|}\Big)\Big|\Big\} \leq C\mu^n.$$

Proof. For any x and y in I, let $x_i = f^{\circ i}(x)$ and $y_i = f^{\circ i}(y)$. From **(4)** of Definition 3, there are constants $C_1 > 0$ and $0 < \mu_1 < 1$ such that

$$|x_i - y_i| \leq C_1\mu_1^{n+m-i}$$

for $0 \leq i < m$. Let $a_0 = \min_{x\in J\in\eta}|f'(x)|$ and let

$$b_0 = \sup_{J\in\eta, x\neq y\in J}\frac{|f'(x) - f'(y)|}{|x-y|^{\alpha}}.$$

Since f is non-critical and $C^{1+\alpha}$, we have $0 < a_0, b_0 < \infty$. Applying Lemma 1 and the chain rule, there are constants $C > 0$ and $0 < \mu < 1$ such that

$$\Big|\log\Big(\frac{|(f^{\circ m})'(x)|}{|(f^{\circ m})'(y)|}\Big)\Big| \leq C\mu^n.$$

∎

Proof of Theorem 1. Suppose $a^* = \cdots w_n$ is a point in Σ^*, where w_n is the first n digits of a^* starting from the right. For any $n, m > 0$, define $I_{w_n} = f^{\circ m}(I_{w_{n+m}})$ and $I_{\sigma^*(w_n)} = f^{\circ m}(I_{\sigma^*(w_{n+m})})$. Hence

$$|s(w_{n+m}) - s(w_n)| = \Big|\frac{(f^{\circ m})'(x)}{(f^{\circ m})'(y)} - 1\Big|\cdot s(w_n)$$

$$= \Big|\frac{(f^{\circ m})'(x)}{(f^{\circ m})'(y)} - 1\Big|\cdot\Big|\frac{(f^{\circ n})'(x')}{(f^{\circ n})'(y')}\Big|\cdot s(w_1)$$

where x and y are in $I_{w_{n+m}}$ and x' and y' are in I_{w_n}. From Lemma 2, there are constants $C > 0$ and $0 < \mu < 1$ such that

$$\Big|\frac{(f^{\circ m})'(x)}{(f^{\circ m})'(y)} - 1\Big| \leq C\mu^n \quad\text{and}\quad \Big|\frac{(f^{\circ n})'(x')}{(f^{\circ n})'(y')}\Big|\cdot s(w_1) \leq C.$$

Thus $|s(w_{n+m}) - s(w_n)| \leq C^2\mu^n$ for all $m > 0$, whence $\{s(w_n)\}_{n=0}^{\infty}$ is a Cauchy sequence. This implies that the scaling function $s(a^*) = \lim_{n\to\infty} s(w_n)$ exists.

To prove that the scaling function s on Σ^* is Hölder, we consider two points $a^* = \cdots w_{n+m}$ and $b^* = \cdots w'_{n+m}$ whose first n digits are the same; that is, $w_n = w'_n$. Following the previous argument,

$$|s(w_{n+m}) - s(w'_{n+m})| = \Big|\frac{(f^{\circ m})'(x)}{(f^{\circ m})'(y)} - \frac{(f^{\circ m})'(x')}{(f^{\circ m})'(y')}\Big| \cdot s(w_n)$$
$$= \Big|\frac{(f^{\circ m})'(x)}{(f^{\circ m})'(y)} - \frac{(f^{\circ m})'(x')}{(f^{\circ m})'(y')}\Big| \cdot \Big|\frac{(f^{\circ n})'(x'')}{(f^{\circ n})'(y'')}\Big| \cdot s(w_1)$$

where x and y are in $I_{w_{n+m}}$, x' and y' are in $I_{w'_{n+m}}$, and x'' and y'' are in I_{w_n}. Hence

$$|s(w_{n+m}) - s(w'_{n+m})| \leq 2C^2\mu^n.$$

As m goes to infinity, we have

$$|s(a^*) - s(b^*)| \leq 2C^2\mu^n;$$

that is, s is Hölder continuous. ∎

3.3. **A critical geometrically finite one-dimensional map.** Suppose f is a critical geometrically finite one-dimensional map. Let $CP = \{c_1, \cdots, c_d\}$ be the set of critical points of f and let $\Gamma = \{\gamma_1, \cdots, \gamma_d\}$ be the set of corresponding exponents.

Theorem 2. *Let f be a critical geometrically finite one-dimensional map. Then the scaling function s, on Σ^*, of f exists.*

Furthermore, because of the existence of critical points for a critical geometrically finite one-dimensional map, we have

Corollary 1. *Let f be a critical geometrically finite one-dimensional map. Then its scaling function s is discontinuous on Σ^*.*

We note that since $\inf_{I\in\eta, x\in I} |f'(x)| = 0$ for a critical geometrically finite one-dimensional map, we can not applying Lemma 1 directly. To overcome this difficulty, we first prove two distortion lemmas. The integer n_0 and the sets U and V are the same as in §3.1. For a point x in M, let $x_i = f^{\circ i}(x)$.

Lemma 3. *There are constants $C > 0$ and $0 < \mu < 1$ such that for any integers $n \geq n_0$ and $m > 0$, and for any points x and y in an interval of η_{n+m}, if x_i and y_i are in V for all $0 \leq i < m$, then*

$$\Big|\log\Big(\frac{|(f^{\circ m})'(x)|}{|(f^{\circ m})'(y)|}\Big)\Big| \leq C\mu^n.$$

Proof. Suppose f is $C^{1+\alpha}$. Let $a_0 = \min_{x\in V} |f'(x)|$ and

$$b_0 = \sup_{x\neq y\in V} \frac{|f'(x) - f'(y)|}{|x-y|^\alpha}.$$

Then $0 < a_0, b_0 < \infty$. From Lemma 1,

$$\Big|\log\Big(\frac{|(f^{\circ m})'(x)|}{|(f^{\circ m})'(y)|}\Big)\Big| \leq \frac{b_0}{a_0}\sum_{i=0}^{m-1} |x_i - y_i|^\alpha.$$

Since x and y are in an interval in η_{n+m}, x_i and y_i are in an interval in η_{n+m-i}. Hence $|x_i - y_i| \leq C\mu^{n+m-i}$ because of **(4)** of Definition 3. Hence there are constants $C > 0$ and $0 < \mu < 1$ such that

$$\Big|\log\Big(\frac{|(f^{\circ m})'(x)|}{|(f^{\circ m})'(y)|}\Big)\Big| \leq C\mu^n.$$

∎

The next lemma is one of the key lemmas in this paper. To present a clear idea of the proof, we first prove it under the assumption that $PCO \cap CP = \emptyset$. For a point x in M, let $x_i = f^{\circ i}(x)$.

Lemma 4. *There are constants $C > 0$ and $0 < \mu < 1$ such that for any integers $n \geq n_0$ and $m > 0$ and any points x and y in an interval of η_{n+m}, if x_m and y_m are in U, then*

$$\Big|\log\Big(\frac{|(f^{\circ m})'(x)|}{|(f^{\circ m})'(y)|}\Big)\Big| \leq C\mu^n.$$

Proof. The ratio $|(f^{\circ m})'(x)|/|(f^{\circ m})'(y)|$ equals the product $\prod_{i=0}^{m-1} |f'(x_i)|/|f'(y_i)|$. We divide this product into two sub-products,

$$\prod_{x_i, y_i \in U} \frac{|f'(x_i)|}{|f'(y_i)|} \quad \text{and} \quad \prod_{x_i, y_i \in V} \frac{|f'(x_i)|}{|f'(y_i)|}.$$

Following the proof of Lemma 3, there are constants C_1, $C_1' > 0$ and $0 < \mu_1 < 1$ such that

$$\Big|\log\Big(\prod_{x_i, y_i \in V} \frac{|f'(x_i)|}{|f'(y_i)|}\Big)\Big| \leq C_1 \sum_{x_i, y_i \in V} |x_i - y_i|^{\alpha} \leq C_1' \mu_1^n.$$

To estimate the first product $\prod_{x_i, y_i \in U} |f'(x_i)|/|f'(y_i)|$, we write it as the product of three factors:

$$\mathcal{I} = \prod_{x_i, y_i \in U} \Big(\frac{|x_i - c_{k_i}|^{\gamma_{k_i}}}{|f(x_i) - f(c_{k_i})|} \frac{|f(y_i) - f(c_{k_i})|}{|y_i - c_{k_i}|^{\gamma_{k_i}}}\Big)^{t_{k_i}},$$

$$\mathcal{II} = \prod_{x_i, y_i \in U} \frac{|y_i - c_{k_i}|^{\gamma_{k_i} - 1}}{|f'(y_i)|} \frac{|f'(x_i)|}{|x_i - c_{k_i}|^{\gamma_{k_i} - 1}},$$

and

$$\mathcal{III} = \prod_{x_i, y_i \in U} \Big(\frac{|f(x_i) - f(c_{k_i})|}{|f(y_i) - f(c_{k_i})|}\Big)^{t_{k_i}},$$

where x_i and y_i are in U_{k_i} and $t_{k_i} = (\gamma_{k_i} - 1)/\gamma_{k_i}$. Applying Lemma 1 and **(1)** of Definition 3, and following the proof of Lemma 1, there are constants $C_2, C_2' > 0$ and $0 < \mu_2 < 1$ such that

$$\Big|\log \mathcal{I}\Big|, \Big|\log \mathcal{II}\Big| \leq C_2 \sum_{x_i, y_i \in U} |x_i - y_i|^{\alpha} \leq C_2' \mu_2^n.$$

Now the proof of Lemma 4 concentrates on the estimate of $\mathcal{III}$. Let

$$\frac{f(x_i) - f(c_{k_i})}{f(y_i) - f(c_{k_i})} = 1 + \frac{f(x_i) - f(y_i)}{f(y_i) - f(c_{k_i})}.$$

Then

$$\mathcal{III} = \exp\Big(\sum_{s=1}^{r-1} \frac{1}{t_{k_{i_s}}} \log\Big|1 + \frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{k_{i_s}})|}\Big|\Big)$$

where $i_1 < i_2 < \cdots < i_{r-1} < m$. Let $i_r = m$. For each i_s, $1 \leq s < r$, consider the interval L_s bounded by y_{i_s} and $c_{k_{i_s}}$ and the map $h_s = f^{\circ(i_{s+1} - i_s)}$. Let $R_s \subseteq L_s$ be the maximal interval containing y_{i_s} such that h_s on R_s is injective. One of the endpoints of R_s is y_{i_s} and the other is a preimage e of a critical point c_{j_s} in CP under $f^{\circ k_s}$ for some $0 \leq k_s < i_{s+1} - i_s$. Let $l_s = i_{s+1} - i_s - k_s$. Then h_s on the minimal interval J_s containing x_{i_s} and R_s is injective and maps J_s onto an interval containing the points $y_{i_{s+1}}$, $x_{i_{s+1}}$ and $f^{\circ l_s}(c_{j_s})$. We enlarge every interval J of V into a closed interval $J' \supset J$ such that $J' \cap CP = \emptyset$ and such that the length of $J' \cap U$ is greater than a constant $a > 0$. Let $V' = \bigcup_{J \in V} J'$ be the union of all these enlarged intervals and let $U' = M \setminus V'$. If $f^{\circ i}(J_s) \subseteq V'$ for all $1 \leq i < i_{s+1} - i_s$, by following the proof of Lemma 3, there is a constant $C_3 > 0$ such that

$$\frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{k_{i_s}})|} \leq C_3 \frac{|x_{i_{s+1}} - y_{i_{s+1}}|}{|y_{i_{s+1}} - f^{\circ l_s}(c_{j_s})|}.$$

Since $y_{i_{s+1}}$ is in U and $f^{\circ l_s}(c_{j_s})$ is in PCO,

$$\frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{k_{i_s}})|} \leq C_3 \frac{|x_{i_{s+1}} - y_{i_{s+1}}|}{D}$$

where $D > 0$ is the distance between U and the post-critical orbits PCO. Otherwise, let $0 < k < i_{s+1} - i_s$ be the smallest integer such that $f^{\circ k}(J_s) \cap U' \neq \emptyset$. Since $f^{\circ i}(J_s) \subseteq V'$ for all $1 \leq i < k$, following the proof of Lemma 3, there is a constant $C_4 > 0$ such that

$$\frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{k_{i_s}})|} \leq C_4 \frac{|x_{i_s+k} - y_{i_s+k}|}{|y_{i_s+k} - f^{\circ k}(e)|}.$$

Since y_{i_s+k} is in V and $f^{\circ k}(e)$ is in U',

$$\frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{k_{i_s}})|} \leq C_4 \frac{|x_{i_s+k} - y_{i_s+k}|}{D'}$$

where $D' > 0$ is the distance between V and U'. Hence, there are constants $C_5 > 0$, $C_5' > 0$ and $0 < \mu_4 < 1$ such that

$$\Big|\log \mathcal{III}\Big| \leq C_5 \sum_{i=0}^{m-1} |x_i - y_i| \leq C_5' \mu_4^n.$$

Combining all the estimates, we have constants $C > 0$ and $0 < \mu < 1$ satisfying the lemma. ∎

If $PCO \cap CP \neq \emptyset$, we need to consider a critical chain: a subset $\{c_{i_1}, \cdots, c_{i_l}\}$ of CP is a critical chain if $f^{\circ j_k}(c_{i_k}) = c_{i_{k+1}}$ for some integers $j_k > 0$ and $\{f^{\circ j}(c_{i_k})\}_{j=1}^{j_k - 1} \cap CP = \emptyset$.

Let x and y be two points in an interval in η_{n+m}, where $n \geq n_0$ and $m > 0$. Let $x_i = f^{\circ i}(x)$ and $y_i = f^{\circ i}(y)$ for $0 \leq i \leq m$. Suppose x_m and y_m are in U. Using the same notation as in the proof of Lemma 4, we consider

$$\mathcal{III} = \exp\Big(\sum_{s=1}^{r-1} \frac{1}{t_{k_{i_s}}} \log\Big|1 + \frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{k_{i_s}})|}\Big|\Big)$$

where $i_1 < i_2 < \cdots < i_{r-1} < m$ and x_{i_s} and y_{i_s} are in $U_{k_{i_s}}$. (Note that $c_{k_{i_s}} \in U_{k_{i_s}}$.) Let $i_r = m$ and denote $c(s) = c_{k_{i_s}}$ for $1 \leq s \leq r$. We divide $\{c(s)\}_{s=1}^m$ into maximal critical chains $\mathcal{I}_1 = \{c(1), \cdots, c(s_1)\}$, $\mathcal{I}_2 = \{c(s_1+1), \cdots, c(s_2)\}$, $\cdots$, $\mathcal{I}_q = \{c(s_{q-1}+1), \cdots, c(s_q)\}$. Remember that $c(s_q) = c_{k_m}$. The number of points in every maximal critical chain is less than or equal to the number of points in CP. Now we can generalize Lemma 4.

Lemma 4′. *There are constants $C > 0$ and $0 < \mu < 1$ such that for any integers $n \geq n_0$ and $m > 0$ and for any points x and y in an interval of η_{n+m}, if x_m and*

y_m are in U and if the last critical chain $\mathcal{I}_q$ contains only one critical point c_{k_m}, then

$$\Big|\log\Big(\frac{|(f^{\circ m})'(x)|}{|(f^{\circ m})'(y)|}\Big)\Big| \leq C\mu^n.$$

Proof. We use the same notation as in the proof of Lemma 4. The estimates of $\mathcal{I}$ and $\mathcal{II}$ are the same as in that proof. Here we add maximal critical chains in the estimates of $\mathcal{III}$. Consider a critical point $c(l)$ in a maximal critical chain $\mathcal{I}_i = \{c(s_{j-1}+1), \cdots, c(s_j)\}$ for $1 \leq j < q$. By arguments similar to those in the proof of Lemma 4 and by the fact that f is comparable to $B_\pm|x-c(l)|^{\gamma(l)} + f(c(l))$ near $c(l)$, there is a positive constant C_6 such that

$$\frac{|f(x_{i_l}) - f(y_{i_l})|}{|f(y_{i_l}) - f(c_{k_{i_l}})|} \leq C_6 \frac{|x_{i_{s_j+1}} - y_{i_{s_j+1}}|^{\frac{1}{\tau_j}}}{D''}$$

where $\tau_j = \prod_{l=s_{j-1}+1}^{s_j} \gamma(l) \leq \gamma = \prod_{i=1}^{d} \gamma_i$ and where D'' is a constant. Therefore, there are constants $C_5 > 0$ and $0 < \mu_5 < 1$ such that

$$\Big|\log \mathcal{III}\Big| \leq C_5\mu_5^n.$$

Combining the estimates for $\mathcal{I}$ and $\mathcal{II}$ together, there are constants $C > 0$ and $0 < \mu < 1$ satisfying the lemma. ∎

Proof of Theorem 2. Suppose $a^* = \ldots w_n$ is a point in Σ^* and $v_n = \sigma^*(w_n)$. Then $w_n = v_n i_0$. We discuss the sequence $\{I_{v_n}\}_{n=1}^\infty$ in two cases: (1) there is an integer $n_1 \geq n_0$ such that $n \geq n_1$ implies $I_{v_n} \subset V$, (2) there is an increasing subsequence $\{n_k\}_{k=1}^\infty$ of integers such that $I_{v_{n_k}} \subseteq U$ for all n_k.

In (1), by arguments similar to those in the proof of Theorem 1, we can prove that $\{s(w_n)\}_{n=1}^\infty$ is a Cauchy sequence. Hence the limit $s(a^*) = \lim_{n\mapsto\infty} s(w_n)$ exists.

In (2), consider all $I_{m_k} \subseteq U_{m_k} \subset U$ and $\{c_{m_k}\}_{k=1}^\infty$. We can find an increasing sequence $\{m_{k_i}\}_{i=1}^\infty$ such that the last critical chain of $\{c_{m_k}\}_{k=k_i}^\infty$ contains only one critical point for every $1 \leq i < \infty$. Let $n_i = m_{k_i}$. For any $n \geq n_i$, the intervals $I_{w_{n_i}}$ and $I_{v_{n_i}}$ are the images of I_{w_n} and I_{v_n} under $f^{\circ(n-n_i)}$, and the intervals $I_{w_{n_1}}$ and $I_{v_{n_1}}$ are the images of $I_{w_{n_i}}$ and $I_{v_{n_i}}$ under $f^{\circ(n_i-n_1)}$. Thus

$$|s(w_n) - s(w_{n_i})| = \Big|\frac{(f^{\circ(n-n_i)})'(x)}{(f^{\circ(n-n_i)})'(y)} - 1\Big| \cdot \frac{|I_{w_{n_i}}|}{|I_{v_{n_i}}|}$$

$$= \Big|\frac{(f^{\circ(n-n_i)})'(x)}{(f^{\circ(n-n_i)})'(y)} - 1\Big| \cdot \Big|\frac{(f^{\circ(n_i-n_1)})'(x')}{(f^{\circ(n_i-n_1)})'(y')}\Big| \cdot s(w_1)$$

where x and y are in I_{v_n} and x' and y' are in $I_{v_{n_i}}$. By Lemma 4′, there are constants $C_1 > 0$ and $0 < \mu_1 < 1$ such that

$$\Big|\frac{(f^{\circ(n-n_i)})'(x)}{(f^{\circ(n-n_i)})'(y)} - 1\Big| \leq C\mu^{n_i} \quad \text{and} \quad \Big|\frac{(f^{\circ(n_i-n_1)})'(x')}{(f^{\circ(n_i-n_1)})'(y')}\Big| \cdot s(w_1) \leq C$$

for $n \geq n_i$. Thus

$$|s(w_n) - s(w_{n_i})| \leq C^2\mu^{n_i}$$

for $n \geq n_i$. The sequence $\{s(w_n)\}_{n=1}^\infty$ is thus a Cauchy sequence and $\lim_{n\to\infty} s(w_n) = s(a^*)$ exists. ∎

Proof of Corollary 1. Suppose c is a critical point of f. It is not periodic and its orbit $\{f^{\circ n}(c)\}_{n=0}^{\infty}$ is finite. There is a periodic point $p \neq c$ and an integer $l > 0$ such that $f^{\circ i}(c) \neq p$ for $0 \leq i < l$ and $f^{\circ l}(c) = p$. Let $a^* = w_m^{\infty}$ be a point in Σ^* such that $\{p\} = \bigcap_{i=1}^{\infty} I_{w_m^i}$ where m is the period of p. From **(4)** of Definition 3 and from Proposition 3, we have $|E_p| = |(f^{\circ m})'(p)| > 1$ and the existence of an integer $0 \leq m_0 < m$ such that for $c^* = (\sigma^*)^{\circ m_0}(a^*)$, $0 < s(c^*) < 1$. We will prove that s is discontinuous at c^*.

Suppose $c^* = u_m^{\infty}$. Without loss of generality, we assume that $\{f^{\circ i}(c)\}_{i=1}^{m-1}$ contains no critical points and that c is not the image of any critical point under any iterate of f. For any u_m^i, there is a $b^* = \cdots v_n u_m^i$ in Σ^* such that $I = I_{v_n u_m^i}$ contains c. For large $i > 0$, I is contained in U and $f^{\circ j}(I)$ is contained in V for every $0 < j \leq n$. Since $f|I$ is asymptotically $B_{\pm}|x-c|^{\gamma} + f(c)$ as n tends to infinity where $\gamma > 1$, Lemma 3 implies that,

$$\lim_{i\to\infty} s(v_n u_m^i) = \lim_{i\mapsto\infty} \Big(s(u_m^i)\Big)^{\gamma} = \Big(s(c^*)\Big)^{\gamma}.$$

From Lemma 4′ and from the proof of Theorem 2, $|s(b^*) - s(v_n w_m^i)| \leq C\mu^{n+m^i}$. We have

$$\lim_{b^*\to c^*} s(b^*) = \Big(s(c^*)\Big)^{\gamma}.$$

Hence c^* is a jump discontinuity of s. ∎

Remark 3. From the proof of Corollary 1, we can actually find out all continuous points and discontinuous points of the scaling function of a critical geometrically finite one-dimensional map. A point $a^* = \cdots w_n$ in Σ_f^* is said to be recurrent if there is an increasing subsequence $\{n_i\}_{i=1}^{\infty}$ of integers such that $I_{w_{n_i}} \subseteq U$ for every n_i. It is said to be non-recurrent if there is an integer $n > 0$ such that all preimages of I_{w_n} under iterates of f are contained in V. The point a^* is said to be wandering if there is an integer $m > 0$ such that $I_{w_n} \subseteq V$ for all $n \geq m$, and if for every integer $n \geq m$, there is another integer $\overline{n}$ such that there is an interval in the preimage of I_{w_n} under $f^{\circ\overline{n}}$ contained in U. We can prove that s on Σ^* is continuous at all recurrent a^*, all non-recurrent a^*, and all wandering points a^* with $s(a^*) = 1$, and that s is discontinuous at wandering points a^* with $0 < s(a^*) < 1$. All discontinuities are jump discontinuities.

Remark 4. From the proof of Corollary 1, if $PCO \cap CP = \emptyset$, then the exponent $\gamma > 1$ of f at a critical point c can be calculated from the scaling function s. Hence the exponent γ is a C^1-invariant, by Proposition 4.

3.4. **Complete smooth invariants.** A geometrically finite one-dimensional map f is simple if $PCO \cap CP = \emptyset$. To avoid notational complication, we consider only simple geometrically finite one-dimensional maps in this section. Let $\mathcal{F}$ be an (orientation-preserving) topological conjugacy class in the space of simple geometrically finite one-dimensional maps, i.e., $\mathcal{F}$ is a subset of simple geometrically finite one-dimensional maps such that every pair of maps f and g in $\mathcal{F}$ are topologically conjugate by an orientation-preserving homeomorphism h. Since all maps f in $\mathcal{F}$ have the same dual phase space Σ^* we may speak of the dual phase space Σ^* of a topological conjugacy class $\mathcal{F}$.

Remark 5. Take a fixed orientation-reversing homeomorphism h_0 of M. If f and g are topologically conjugate by an orientation-reversing homeomorphism h, then

f and $\tilde{g} = h_0 \circ g \circ h_0^{-1}$ are topologically conjugate by an orientation-preserving homeomorphism.

Suppose f and g are in $\mathcal{F}$ and h is the conjugacy from f to g, i.e., $h \circ f = g \circ h$. In the paper [7], we proved that h must be a quasisymmetric homeomorphism [1]. This implies that h is α-Hölder continuous for some $0 < \alpha \leq 1$ [1]. Usually, h is not Lipschitz because f has a lot of Lipschitz invariants, for example, all eigenvalues of f at periodic points. However, we have

Theorem 3. *Let f and g be maps in $\mathcal{F}$ and let h is the conjugacy from f to g. Then h is bi-Lipschitz continuous if the scaling functions s_f and s_g on Σ^* of f and g are the same.*

Proof. Let η be the natural Markov partition of M to f. Let η_n be the n^{th} partition induced by f with η. Let m_0 be the number of the intervals in η and define $k_0 = m_0 + n_0$, where n_0 is the fixed integer in §3.1. For any integer $n > 0$ and any interval $I_{w_{k_0} w_n}$ in η_{n+k_0},

$$\frac{|h(I_{w_{k_0} w_n})|}{|I_{w_{k_0} w_n}|} = \frac{|s_g(w_{k_0} w_n)|}{|s_f(w_{k_0} w_n)|} \cdot \frac{|h(I_{w_{k_0} v_n})|}{|I_{w_{k_0} v_n}|}$$

where $w_{k_0} v_n = \sigma^*(w_{k_0} w_n)$.

Let $a^* = \cdots u_m w_{k_0} w_n$ be a point in Σ^*. For the sequence $\{I_{u_m w_{k_0} w_n}\}_{m=0}^{\infty}$ we consider three cases: **(1)** $I_{w_{k_0} w_n}$ is contained in U; **(2)** $I_{u_m w_{k_0} w_n}$ is contained in V for every $m \geq 0$; **(3)** there is an integer $m \geq 1$ such that $I_{u_i w_{k_0} w_n}$ is contained in V for every $0 \leq i \leq m$ and $I_{u_{m+1} w_{k_0} w_n}$ is contained in U.

In case **(1)**, we use Lemma 4 and Lemma 3 for both f and g to find constants $C_1 > 0$ and $0 < \mu_1 < 1$ such that

$$\left|\log\left(\frac{s_f(a^*)}{s_f(w_{k_0} w_n)}\right)\right| \leq C_1 \mu_1^n$$

and

$$\left|\log\left(\frac{s_g(a^*)}{s_g(w_{k_0} w_n)}\right)\right| \leq C_1 \mu_1^n.$$

Because $s_g = s_f$, for $C_2 = 2C_1$ and $\mu_2 = \mu_1$,

$$\left|\log\left(\frac{s_f(w_{k_0} w_n)}{s_g(w_{k_0} w_n)}\right)\right| \leq C_2 \mu_2^n.$$

In case **(3)**, suppose I is the interval in η_{k_0} having a critical point c of f as an endpoint and containing $I_{u_{m+1} w_{k_0} w_n}$. Then there is an integer $0 < l < m_0$ such that $f^{\circ l}(c)$ is a periodic point p of f and $f^{\circ l}(I_{u_{m+1} w_{k_0} w_n}) \subset f^{\circ l}(I)$. We note that $f^{\circ l}(I)$ is an interval in $\eta_{k_0 - l}$ and that $f^{\circ l}(I_{u_{m+1} w_{k_0} w_n})$ is an interval in η_{m+1+k_0+n-l}. Remember that p is an endpoint of $f^{\circ l}(I)$. We can now find a point $b^* = \cdots w'_j$ in Σ^* such that the first $m + 1 + k_0 + n - l$ digits of a^* and b^* (starting from the right) are the same and such that the interval $I_{w'_j}$ in η_j is contained in V for every $j > 0$ and tends to the periodic orbit $\bigcup_{i=0}^{\infty} f^{\circ i}(p)$ as j goes to infinity. Thus we get another sequence $\{I_{w'_j}\}$ which is in case **(2)**. If $m + 1 - l > 0$, then $I_{w_{k_0} w_n}$ is in $\{I_{w'_j}\}$. If $m + 1 - l < 0$, then we have that $I_{w_{k_0 - l} w_n}$ is in $\{I_{w'_j}\}$; from Lemma 3, there are now constants $C_3 > 0$ and $0 < \mu_3 < 1$ such that

$$\left|\log\left(\frac{s_f(w_{k_0} w_n)}{s_f(w_{k_0 - l} w_n)}\right)\right| \leq C_3 \mu_3^n.$$

Only case **(2)** remains. By applying Lemma 3, there are constants $C_4 > 0$ and $0 < \mu_4 < 1$ such that

$$\Big|\log\Big(\frac{s_f(b^*)}{s_f(w_{k_0}w_n)}\Big)\Big| \leq C_4\mu_4^n$$

and

$$\Big|\log\Big(\frac{s_g(b^*)}{s_g(w_{k_0}w_n)}\Big)\Big| \leq C_4\mu_4^n.$$

Because $s_g = s_f$, for $C_5 = 2C_4$ and $\mu_5 = \mu_4$,

$$\Big|\log\Big(\frac{s_f(w_{k_0}w_n)}{s_g(w_{k_0}w_n)}\Big)\Big| \leq C_5\mu_5^n.$$

Let C_0 be the minimum of the ratios $\log(|h(I_w)|/|I_w|)$ for I_w in η_{k_0}. From the arguments above, there are constants $C > 0$ and $0 < \mu < 1$ such that

$$\Big|\log\frac{|h(I_{w_{k_0}w_n})|}{|I_{w_{k_0}w_n}|}\Big| \leq C_0 + C\sum_{i=1}^{n}\mu^i \leq C_0 + \frac{C\mu}{1-\mu}$$

for all $I_{w_{k_0}w_n}$ in η_{k_0+n} and all $n \geq 0$. Hence there is a constant, still denoted $C > 0$, such that

$$C^{-1} \leq \frac{|h(I_{w_{k_0}w_n})|}{|I_{w_{k_0}w_n}|} \leq C$$

for all $I_{w_{k_0}w_n}$ in η_{n+k_0} and all $n \geq 0$. Since the union of boundary points of all intervals in η_{n+k_0} for all $n \geq 0$ is dense in M,

$$C^{-1} \leq \frac{|h(x)-h(y)|}{|x-y|} \leq C$$

for every pair x and y in M. In other words, h is bi-Lipschitz continuous. ∎

A geometrically finite one-dimensional map f is mixing if for any interval I in η_f, there is an integer $n > 0$ such that $f^{\circ n}(I) = M$. Since the mixing condition is topologically invariant, we may speak of a topological conjugacy class $\mathcal{F}$ being mixing.

A singular point p of f is a fold singular point if $f'(x)f'(2p-x) < 0$ for $x \neq p$ near p. Let FSP denote the set of all fold singular points of a geometrically finite one-dimensional map f and let $FSO = \bigcup_{i=1}^{\infty} f^{\circ i}(FSP)$ denote the post fold singular orbits. Let ξ be the set of closures of intervals of $M \setminus FSO$.

Let f be a geometrically finite one-dimensional map. Let η be the natural Markov partition of M to f. Let η_n be the n^{th} partition of M induced by f with η. Let $GSO = \bigcup_{i=0}^{\infty} \cup_{J\in\eta_n} \partial J$ be the grand singular orbits of f. For a mixing topological conjugacy class $\mathcal{F}$, we have the following rigidity result.

Theorem 4. *Let f and g be maps in a mixing topological conjugacy class $\mathcal{F}$. Let h be the conjugacy from f to g, i.e., $h \circ f = g \circ h$. Then $h|I$ for any I in ξ is a $C^{1+\beta}$-diffeomorphism for some $0 < \beta \leq 1$ if and only if (i) h is differentiable at one point p in $M \setminus GSO$ with non-zero derivative and (ii) the exponents and asymmetries of f and g at all corresponding critical points are the same.*

We prove Theorem 4 by means of several lemmas. Let $\mathcal{F}$ be a mixing topological conjugacy class in the space of simple geometrically finite one-dimensional maps and let f and g be maps in $\mathcal{F}$. Suppose both f and g are $C^{1+\alpha}$ for some $0 < \alpha \leq 1$. Let h be the conjugacy from f and g: $h \circ f = g \circ h$. Let η be the natural Markov partition of M to f. Let η_n be the n^{th} partition of M induced by f with η for

$0 \leq n < \infty$. Let Σ^* be the dual phase space of $\mathcal{F}$. Let $CP = \{c_1, \cdots, c_d\}$ be the critical points of f. Let $\Gamma = \{\gamma_1, \cdots, \gamma_d\}$ be the set of corresponding exponents of f at critical points and let $AS = \{A_1, A_2, \cdots, A_d\}$ be the set of corresponding asymmetries of f. Let n_0 and U and V be the fixed integer and the sets in §3.1 for f.

Suppose $I = I_{w_{n_0}} \in \eta_{n_0}$ and $a^* = \cdots w_n w_{n_0}$ is a point in Σ^*. Then $I_{w_n w_{n_0}}$ is an interval in η_{n+n_0}. For any x and y in I, let x_n and y_n in $I_{w_n w_{n_0}}$ be the preimage of x and y under $f^{\circ n}$ for $n \geq 0$.

Lemma 5. *There is a constant $C > 0$ such that if all $I_{w_n w_{n_0}}$ are in V, then*

$$\Big|\log\Big(\prod_{n=1}^{\infty} \frac{|f'(y_n)|}{|f'(x_n)|}\Big)\Big| \leq C \cdot |x-y|^{\alpha}$$

and

$$\Big|\log\Big(\prod_{n=1}^{\infty} \frac{|g'(h(x_n))|}{|g'(h(y_n))|}\Big)\Big| \leq C \cdot |h(x)-h(y)|^{\alpha}.$$

Proof. From the proof of Lemma 3, there is a constant $C_1 > 0$ such that

$$\Big|\log\Big(\prod_{i=1}^{\infty} \frac{|f'(y_i)|}{|f'(x_i)|}\Big)\Big| \leq C_1 \cdot \sum_{i=1}^{\infty} |x_i - y_i|^{\alpha}.$$

By applying Lemma 3, we have a constant $C_2 > 0$ such that

$$\frac{|x_n - y_n|}{|I_{w_{n+n_0}}|} \leq C_2 \cdot \frac{|x-y|}{|I_{w_{n_0}}|}$$

for all $1 \leq n < \infty$. So there are constants $C_3 > 0$ and $0 < \mu_1 < 1$ such that

$$|x_n - y_n| \leq C_3 \cdot \mu_1^n \cdot |x-y|$$

for all $0 \leq n < \infty$. This implies that

$$\Big|\log\Big(\prod_{n=1}^{\infty} \frac{|f'(y_n)|}{|f'(x_n)|}\Big)\Big| \leq C \cdot |x-y|^{\alpha}$$

for some constant $C > 0$. Similarly,

$$\Big|\log\Big(\prod_{n=1}^{\infty} \frac{|g'(h(y_n))|}{|g'(h(x_n))|}\Big)\Big| \leq C \cdot |h(x)-h(y)|^{\alpha}.$$

∎

Lemma 6. *There is a constant $C > 0$ such that if $I_{w_{n_0}}$ is in U, then*

$$\Big|\log\Big(\prod_{n=1}^{\infty} \frac{|f'(y_n)|}{|f'(x_n)|}\Big)\Big| \leq C \cdot |x-y|^{\alpha}$$

and

$$\Big|\log\Big(\prod_{n=1}^{\infty} \frac{|g'(h(x_n))|}{|g'(h(y_n))|}\Big)\Big| \leq C \cdot |h(x)-h(y)|^{\alpha}.$$

Proof. From the proof of Lemma 4, there is a constant $C_1 > 0$ such that

$$\Big|\log\Big(\prod_{n=1}^{\infty}\frac{|f'(y_n)|}{|f'(x_n)|}\Big)\Big| \leq C_1 \cdot \sum_{n=1}^{\infty}|x_n - y_n|^{\alpha}.$$

By applying Lemma 4, we have a constants $C_2 > 0$ such that

$$\frac{|x_n - y_n|}{|I_{w_{n+n_0}}|} \leq C_2 \cdot \frac{|x-y|}{|I_{w_{n_0}}|}$$

for all $1 \leq n < \infty$. So there are constant $C_3 > 0$ and $0 < \mu_1 < 1$ such that

$$|x_n - y_n| \leq C_3 \cdot \mu_1^n \cdot |x-y|$$

for all $0 \leq n < \infty$. This implies that

$$\Big|\log\Big(\prod_{n=1}^{\infty}\frac{|f'(y_n)|}{|f'(x_n)|}\Big)\Big| \leq C \cdot |x-y|^{\alpha}$$

for a constant $C > 0$. Similarly,

$$\Big|\log\Big(\prod_{n=1}^{\infty}\frac{|g'(h(y_n))|}{|g'(h(x_n))|}\Big)\Big| \leq C \cdot |h(x)-h(y)|^{\alpha}.$$

■

Let p be a point in $M \setminus GSO$. Consider the grand backward orbit of p under f, $GO(p) = \bigcup_{i=0}^{\infty} f^{-i}(p)$. Then $GO(p)$ is a dense subset of M since f is mixing. From the equation $h \circ f = g \circ h$, we have that if $h'(f(x)) \neq 0$ exists and if x is not a singular point, then

$$h'(x) = \frac{f'(x)}{g'(h(x))} \cdot h'(f(x)) \neq 0$$

exists too. Therefore, if $h'(p) \neq 0$ exits, then $h'(x) \neq 0$ exists for any $x \in GO(p)$.

Lemma 7. *If h is differentiable at a point p in $M \setminus GSO$ with non-zero derivative, then $h'|GO(p)$ is continuous at p.*

Proof. Since $h'(p) \neq 0$ exists, there is an open interval $p \in W$ such that $h|W$ is bi-Lipschitz. Because the eigenvalue of an expanding periodic point of f is a bi-Lipschitz invariant, we have

$$(f^{\circ n})'(q) = (g^{\circ n})'(h(q))$$

if $f^{\circ n}(q) = q \in W$.

Suppose $p \in I_k = I_{w_k w_{n_0}} \in \eta_{k+n_0}$ and that $I_k \subseteq W$. For any $x \in I_k \cap GO(p)$ with $f^{\circ n}(x) = p$, let $x_i = f^{\circ(n-i)}(x)$. Let $x_i \in I_{i+k} \in \eta_{i+k+n_0}$. Then $I_{n+k} \subseteq f^{\circ n}(I_{n+k}) = I_k$. There is a point $q \in I_{n+k}$ such that $f^{\circ n}(q) = q$. From $h \circ f = g \circ h$, we have

$$\frac{h'(p)}{h'(x)} = \frac{(g^{\circ n})'(h(x))}{(f^{\circ n})'(x)} = \frac{(g^{\circ n})'(h(x))}{(g^{\circ n})'(h(q))} \cdot \frac{(f^{\circ n})'(q)}{(f^{\circ n})'(x)}.$$

Let $q_i = f^{\circ(n-i)}(q)$ and consider

$$\frac{(f^{\circ n})'(q)}{(f^{\circ n})'(x)} = \prod_{i=1}^{n}\frac{f'(q_i)}{f'(x_i)}.$$

If $I_k \subseteq U$, then from Lemma 6,

$$\Big|\log\Big(\frac{|(f^{\circ n})'(q)|}{|(f^{\circ n})'(x)|}\Big)\Big| \le C_1|p-q|^\alpha \le C\mu_1^k$$

where $C_1 > 0$ and $0 < \mu_1 < 1$ are constants. Otherwise, consider the smallest integer $m > 0$ such that $I_{m+k} \subseteq U_j \subset U$. From Lemma 5, there are constants $C_2 > 0$ and $0 < \mu_2 < 1$ such that

$$\Big|\log\Big(\prod_{i=1}^{m-1}\frac{|f'(q_i)|}{|f'(x_i)|}\Big)\Big| \le C_2|p-q|^\alpha \le C_2\mu_2^k.$$

From Lemma 6, there are constants $C_3 > 0$ and $0 < \mu_3 < 1$ such that

$$\Big|\log\Big(\prod_{i=m+1}^{n}\frac{|f'(q_i)|}{|f'(x_i)|}\Big)\Big| \le C_3|x_m-q_m|^\alpha \le C_3\mu_3^{k+m}.$$

Because $f|U_j$ and $f'|U_j$ are comparable to $|z-c_j|^{\gamma_j}+f(c_j)$ and $|z-c_j|^{\gamma_j-1}$, there are constants $C_4 > 0$ and $0 < \mu_4 < 1$ such that

$$\begin{aligned}\Big|\log\Big(\frac{|f'(q_m)|}{|f'(x_m)|}\Big)\Big| &\le C_4|x_m-q_m|^\alpha + \frac{1}{\gamma_j-1}\Big|\log\Big(\frac{|x_m-c_j|}{|q_m-c_j|}\Big)\Big| \\ &\le C_4\mu_4^{m+k} + \frac{1}{\gamma_j-1}\Big|\log\Big(\frac{|x_m-c_j|}{|q_m-c_j|}\Big)\Big|.\end{aligned}$$

If the distance $d_k = \mathrm{dist}(I_k, PCO)$ between I_k and PCO is greater than the length $|I_k|$ of I_k, then $|x_m-q_m| \le C_6\min\{|x_m-c_j|, q_m-c_j|\}$ and

$$\begin{aligned}\Big|\log\Big(\frac{|x_m-c_j|}{|q_m-c_j|}\Big)\Big| &\le C_7\frac{|x_m-q_m|}{|q_m-c_j|} \le C_8\Big(\frac{|x_{m+1}-q_{m+1}|}{|q_{m+1}-f(c_j)|}\Big)^{\frac{1}{\gamma_j}} \\ &\le C_9\Big(\frac{|p-q|}{d_k}\Big)^{\frac{1}{\gamma_j}} \le C_{10}\Big(\frac{\mu_{10}^k}{d_k}\Big)^{\frac{1}{\gamma_j}},\end{aligned}$$

where $C_i > 0$ for $i = 7, 8, 9, 10$ and $0 < \mu_{10} < 1$ are constants. So we have constants $C > 0$ and $0 < \mu < 1$ such that

$$\Big|\log\Big(\frac{|(f^{\circ n})'(q)|}{|(f^{\circ n})'(x)|}\Big)\Big| \le C(\mu^k + \mu^{k+m} + \frac{\mu^k}{d_k}).$$

Similarly, we can get

$$\Big|\log\Big(\frac{|(g^{\circ n})'(h(x))|}{|(g^{\circ n})'(h(q))|}\Big)\Big| \le C(\mu^k + \mu^{k+m} + \frac{\mu^k}{d_k'})$$

where $d_k' = \mathrm{dist}(h(I_k), h(PCO))$. Thus

$$\Big|\log\frac{h'(p)}{h'(x)}\Big| \le 2C(\mu^k + \mu^{k+m} + \frac{\mu^k}{\min\{d_k, d_k'\}}).$$

This implies that $h'|GO(p)$ is continuous at p. ∎

Corollary 2. *If h is differentiable at a point p in $M\setminus GSO$ with nonzero derivative, then $h'|GO(p)$ is continuous at every point x in $GO(p)$.*

Proof. We use the same notation as in Lemma 7. For any $x \in GO(p)$, let $f^{\circ n}(x) = p$, let $p \in I_k = I_{w_k w_{n_0}} \in \eta_{k+n_0}$, and let $x \in I_{n+k} \in \eta_{n+k+n_0}$ such that $f^{\circ n}(I_{n+k}) = I_k$. For any $y \in I_{n+k}$, we have

$$\frac{h'(x)}{h'(y)} = \frac{(f^{\circ n})'(x)}{(f^{\circ n})'(y)} \cdot \frac{(g^{\circ n})'(h(y))}{(g^{\circ n})'(h(x))} \cdot \frac{h'(p)}{h'(f^{\circ n}(y))}.$$

As in the proof of Lemma 7, we have

$$\Big|\log\Big(\frac{h'(x)}{h'(y)}\Big)\Big| \leq C(\mu^k + \mu^{k+m} + \frac{\mu^k}{\min\{d_k, d_k'\}}) + \Big|\log\Big(\frac{h'(p)}{h'(f^{\circ n}(y))}\Big)\Big|.$$

Now Lemma 7 implies that $h'|GO(p)$ is continuous at x. ∎

An interval $I_{w_{n_0}}$ in η_{n_0} is called critical if one of its end-points is a critical point of f. Otherwise, it is called non-critical.

Lemma 8. *If h is differentiable at a point p in $M \setminus GSO$ with non-zero derivative, then the restriction of h to every critical interval in η_{n_0} is $C^{1+\beta}$ for some $0 < \beta \leq 1$.*

Proof. Suppose I is a critical interval in η_{n_0}. Then $I \subset U$. Since f is mixing, there is a preimage I_n of I under $f^{\circ n}$ such that I_n tends to p.

For any x and y in I, let x_n and y_n in I_n be the preimages of x and y under $f^{\circ n}$. From the equation $h \circ f = g \circ h$, we have

$$\frac{h'(x)}{h'(y)} = \frac{(g^{\circ n})'(h(x_n))}{(f^{\circ n})'(x_n)} \cdot \frac{(f^{\circ n})'(y_n)}{(g^{\circ n})'(h(y_n))} \cdot \frac{h'(x_n)}{h'(y_n)}.$$

Thus,

$$\Big|\log\Big(\frac{h'(x)}{h'(y)}\Big)\Big| \leq \Big|\log\Big(\frac{(g^{\circ n})'(h(x_n))}{(g^{\circ n})'(h(y_n))}\Big)\Big| + \Big|\log\Big(\frac{(f^{\circ n})'(y_n)}{(f^{\circ n})'(x_n)}\Big)\Big| + \Big|\log\Big(\frac{h'(x_n)}{h'(y_n)}\Big)\Big| \tag{1}$$

$$\leq C\Big(|x-y|^\alpha + |h(x)-h(y)|^\alpha\Big) + \Big|\log\Big(\frac{h'(x_n)}{h'(y_n)}\Big)\Big|. \tag{2}$$

Since $h'(x_n) \to h'(p)$ and $h'(y_n) \to h'(p)$ as n tends to infinity, we have

$$\Big|\log\Big(\frac{h'(x)}{h'(y)}\Big)\Big| \leq C\Big(|x-y|^\alpha + |h(x)-h(y)|^\alpha\Big).$$

This implies that $h'|(I \cap GO(p))$ is uniformly continuous. Therefore, it can be extended to a continuous function h' on I. Furthermore, the last inequality implies again that h is $C^{1+\beta}$ for $\beta = \alpha$. ∎

Lemma 9. *If h is differentiable at a point p in $M \setminus GSO$ with non-zero derivative and if the exponents of f and g at corresponding critical points are the same, then the restriction of h to every interval $I \subseteq V$ in η_{n_0} is $C^{1+\beta}$ for some $0 < \beta \leq 1$.*

Proof. Suppose I_n in η_{n_0+n} is a preimage of I under $f^{\circ n}$. Since f is mixing, we can choose I_n such that I_n tends to p. Let $I_{n,i} = f^{\circ(n-i)}(I_n)$ for $0 \leq i \leq n$.

For any x and y in I and $n > 0$, let x_n and y_n in I_n be the preimages of x and y under $f^{\circ n}$. From the equation $h \circ f = g \circ h$, we have

$$\frac{h'(x)}{h'(y)} = \frac{(g^{\circ n})'(h(x_n))}{(f^{\circ n})'(x_n)} \cdot \frac{(f^{\circ n})'(y_n)}{(g^{\circ n})'(h(y_n))} \cdot \frac{h'(x_n)}{h'(y_n)}.$$

Thus,

$$\Big|\log\Big(\frac{h'(x)}{h'(y)}\Big)\Big| \leq \Big|\log\Big(\frac{|(g^{\circ n})'(h(x_n))|}{|(g^{\circ n})'(h(y_n))|}\Big)\Big| + \Big|\log\Big(\frac{|(f^{\circ n})'(y_n)|}{|(f^{\circ n})'(x_n)|}\Big)\Big| + \Big|\log\Big(\frac{h'(x_n)}{h'(y_n)}\Big)\Big|.$$

Let $m = m(n) > 0$ be the smallest integer such that $I_{m,n} \subseteq U_j \subset U$. Then

$$\begin{aligned}\Big|\log\Big(\frac{h'(x)}{h'(y)}\Big)\Big| \leq \Big|\sum_{i=1}^{m-1}\Big(\log|f'(y_i)| - \log|f'(x_i)|\Big)\Big| \\ + \Big|\sum_{i=1}^{m-1}\Big(\log|g'(h(x_i))| - \log|g'(h(y_i))|\Big)\Big| \\ + \Big|\log\Big(\frac{|g'(h(x_m))|}{|f'(x_m)|}\cdot\frac{|f'(y_m)|}{|g'(h(y_m))|}\Big)\Big| \\ + \Big|\sum_{i=m+1}^{n}\Big(\log|f'(y_i)| - \log|f'(x_i)|\Big)\Big| \\ + \Big|\sum_{i=m+1}^{n}\Big(\log|g'(h(x_i))| - \log|g'(h(y_i))|\Big)\Big|.\end{aligned}$$

From Lemma 5, there is a constant $C_1 > 0$ such that

$$\Big|\sum_{i=1}^{m-1}\Big(\log|f'(y_i)| - \log|f'(x_i)|\Big)\Big| \leq C_1|x-y|^{\alpha}$$

and

$$\Big|\sum_{i=1}^{m-1}\Big(\log|g'(h(x_i))| - \log|g'(h(y_i))|\Big)\Big| \leq C_1|h(x)-h(y)|^{\alpha}.$$

From Lemma 6, there are constants $C_2 > 0$, $C_3 > 0$, and $C_4 > 0$ such that

$$\begin{aligned}&\Big|\sum_{i=m+1}^{n}\Big(\log|f'(y_i)| - \log|f'(x_i)|\Big)\Big| \\ &\quad \leq C_2|x_m - y_m|^{\alpha} \leq C_3|x_{m-1}-y_{m-1}|^{\frac{\alpha}{\gamma_j}} \leq C_4|x-y|^{\frac{\alpha}{\gamma_j}}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\Big|\sum_{i=m+1}^{n}\Big(\log|g'(h(x_i))| - \log|g'(h(y_i))|\Big)\Big| \\ &\quad \leq C_2|h(x_m)-h(y_m)|^{\frac{\alpha}{\gamma_j}} \leq C_4|h(x)-h(y)|^{\frac{\alpha}{\gamma_j}}.\end{aligned}$$

Now we consider

$$S = \frac{|g'(h(x_m))|}{|f'(x_m)|}\cdot\frac{|f'(y_m)|}{|g'(h(y_m))|}.$$

Define

$$S = S_1 \cdot S_2 \cdot S_3$$

where

$$S_1 = \frac{|g'(h(x_m))|}{|h(x_m)-h(c_j)|^{\gamma_j-1}}\cdot\frac{|h(y_m)-h(c_j)|^{\gamma_j-1}}{|g'(h(y_m))},$$

$$S_2 = \frac{|x_m - c_j|^{\gamma_j-1}}{|f'(x_m)|}\cdot\frac{|f'(y_m)|}{|y_m - c_j|^{\gamma_j-1}},$$

and

$$S_3 = \Big(\frac{|h(x_m) - h(c_j)|}{|x_m - c_j|}\Big)^{\gamma_j - 1} \cdot \Big(\frac{|y_m - c_j|}{|h(y_m) - h(c_j)|}\Big)^{\gamma_j - 1}.$$

Lemma 8 implies that

$$\Big|\log S_3\Big| \le C_5|x_m - y_m|^{\alpha} \le C_6|x_{m-1} - y_{m-1}|^{\frac{\alpha}{\gamma_j}} \le C_7|x - y|^{\frac{\alpha}{\gamma_j}}$$

where C_i for $i = 5$, 6, 7 are constants. From **(1)** of Definition 3,

$$\Big|\log S_2\Big| \le C_8|x_m - y_m|^{\alpha} \le C_9|x_{m-1} - y_{m-1}|^{\frac{\alpha}{\gamma_j}} \le C_{10}|x - y|^{\frac{\alpha}{\gamma_j}}$$

and

$$\Big|\log S_1\Big| \le C_8|h(x_m) - h(y_m)|^{\alpha} \le C_9|h(x_{m-1}) - h(y_{m-1})|^{\frac{\alpha}{\gamma_j}} \le C_{10}|h(x) - h(y)|^{\frac{\alpha}{\gamma_j}}$$

where C_i for $i = 8$, 9, and 10 are constants. Thus,

$$\Big|\log\Big(\frac{h'(x)}{h'(y)}\Big)\Big| \le C(|x - y|^{\frac{\alpha}{\gamma}} + |h(x) - h(y)|^{\frac{\alpha}{\gamma}})$$

where $\gamma = \max\{\gamma_i\}_{i=1}^{d}$. So $h'|(I \cap GO(p))$ is uniformly continuous. It can be extended to a continuous function on I. Furthermore, the last inequality implies that $h|I$ is $C^{1+\beta}$ for $\beta = \alpha/\gamma$. ∎

Proof of Theorem 4. The "only if" part follows from direct calculation. Let us prove the "if" part. From Lemmas 8 and 9, h is $C^{1+\beta}$ for some $0 < \beta \le 1$ when it is restricted to any interval I in η_{n_0}. Using the equation $h \circ f = g \circ h$, we have that h is $C^{1+\beta}$ when it is restricted to any interval I in the natural Markov partition η of M to f. It remains to check that h is continuous on I for any I in ξ. If s is a singular point, then we can find an open interval W contained in an interval I_0 of η_0 such that $s \in f(W)$ and $f : W \to f(W)$ is a diffeomorphism. This implies that h is continuous at s. If s is not a fold singular point, we have, from the equation $h \circ f = g \circ h$, that for $x \ne s$ near s,

$$h'(f(x)) = \frac{g'(h(x))}{f'(x)} h'(x).$$

This implies that

$$h'(f(s)+) = \frac{B_{+,f}}{B_{+,g}} h'(s) \qquad \text{and} \qquad h'(f(s)-) = \frac{B_{-,f}}{B_{-,g}} h'(s)$$

where $B_{\pm,f}$ and $B_{\pm,g}$ are the numbers in the beginning of §3 for f and g. Because f and g have the same asymmetry at s, the last equation implies that

$$h'(f(s)-) = h'(f(s)+).$$

So h is continuous at $f(s)$. Similarly, h is continuous at all points $f^{\circ i}(s)$ for $i \ge 0$ if s is not a fold singular point. Therefore, the restriction of h to any interval I in ξ is $C^{1+\beta}$. ∎

One consequence of Theorems 3 and 4 is that scaling functions together with exponents and asymmetries are complete C^1-invariants as follows:

Theorem 5. *Let f and g be maps in a mixing topological conjugacy class $\mathcal{F}$ and let Σ^* be the dual space of $\mathcal{F}$. Let h be the topological conjugacy from f to g, i.e., $f \circ h = h \circ g$. Then $h|I$ for any I in ξ is a $C^{1+\beta}$-diffeomorphism for some $0 < \beta \leq 1$ if and only if the scaling functions s_f and s_g, on Σ^*, and the exponents and asymmetries of f and g at corresponding critical points are the same.*

Proof. The "only if" part follows from Proposition 4, Remark 2 and direct calculation. Let us prove the "if" part. From Theorem 3, h is bi-Lipschitz. It is differentiable at almost all points in M. A bi-Lipschitz function is absolutely continuous; therefore, $h'(x) > 0$ exists for almost all points in M. Since GSO has measure zero, there is at least one point p in $M \setminus GSO$ such that $h'(p) > 0$ exists. Theorem 5 now follows from Theorem 4. ∎

Remark 6. In [4] and [5] we have studied a special topological conjugacy class, which we call the space of Ulam-von Neumann transformations, of geometrically finite one-dimensional maps. We have shown that the set of eigenvalues at periodic points and the exponent and asymmetry at a unique critical point of a map in this space are complete C^1-invariants. This result can be also discussed for a mixing geometrically finite one-dimensional map.

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